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On quasivector spaces of convex bodies and zonotopes

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In this work the theory of quasivector spaces has been briefly outlined and applied for computation with zonotopes. An approximation problem for zonotopes in the plane has been formulated and an algorithm for its solution has been proposed.

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1. Introduction

The abstract axiomatic study of convex bodies with Minkowski addition and multiplication by scalars leads to the concept of quasilinear space [3,6,7]. A quasilinear space over the field of reals can be defined as an additive Abelian monoid with cancellation law endowed with multiplication by scalars. Every quasilinear space can be embedded in an (additive) group; thereby a natural isomorphic extension of the multiplication by scalars leads to quasilinear spaces with group structure briefly called quasivector spaces [5]. Quasivector spaces obey all axioms of vector spaces, but in place of the second distributive law we have: $(\alpha + \beta) * c = \alpha * c + \beta * c$, if $\alpha\beta \geqslant 0$.

Every quasivector space is a direct sum of a vector space and a symmetric quasivector space [4,5]. On the other side, symmetric quasivector spaces are equivalent to vector spaces in the sense that all algebraic operations in both spaces are mutually representable. This equivalence enables us to transfer basic vector space concepts (such as linear combination, basis, dimension, etc.) to symmetric quasivector spaces. Another important fact is that symmetric quasivector spaces with finite basis are isomorphic to a canonic space similar to $\mathbb{R}^n$, see example 1 below. These results can be used for computations with generalized convex bodies as then we actually work in a vector space. In the present work this has been demonstrated for the special case of zonotopes. In particular, an approximation problem related to zonotopes has been formulated and solved using the theory of quasivector spaces.

In section 2 we briefly introduce some notation and give some properties of quasivector spaces, and, in particular, of symmetric quasivector spaces. Section 3 is devoted to zonotopes, and section 4 – to an approximation problem for centred zonotopes in the plane.

2. Quasivector spaces

By $\mathbb{R}$ we denote the set of reals; we use the same notation for the linearly ordered field of reals $\mathbb{R} = (\mathbb{R}, +, \cdot, \leqslant)$. For any integer $n \geqslant 1$ denote by $\mathbb{R}^n$ the set of all n-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, where $\alpha_i \in \mathbb{R}$. The set $\mathbb{R}^n$ forms a vector space $\mathbb{V}^n = (\mathbb{R}^n, +, \mathbb{R}, \cdot)$ under addition and multiplication by scalars.

Every Abelian monoid $(\mathcal{M}, +)$ with cancellation law induces an Abelian group $(\mathcal{D}(\mathcal{M}), +)$, where $\mathcal{D}(\mathcal{M}) = \mathcal{M}^2/\sim$ consists of all pairs (A, B) factorized by the congruence relation $\sim$: $(A, B) \sim (C, D)$ iff $A + D = B + C$, for all $A, B, C, D \in \mathcal{M}$. Addition in $\mathcal{D}(\mathcal{M})$ is $(A, B) + (C, D) = (A + C, B + D)$. The null element of $\mathcal{D}(\mathcal{M})$ is the class (Z, Z), $Z \in \mathcal{M}$; we have $(Z, Z) \sim (0, 0)$. The opposite element to $(A, B) \in \mathcal{D}(\mathcal{M})$ is $\mathrm{opp}(A, B) = (B, A)$. All elements of $\mathcal{D}(\mathcal{M})$ admitting the form $(A, 0)$ are called *proper* and the remaining are *improper*. The opposite of a proper element is improper; $\mathrm{opp}(A, 0) = (0, A)$ unless $A = 0$.

Definition 1. Let $(\mathcal{M}, +)$ be an Abelian monoid with cancellation law. Assume that a mapping "$*$" (multiplication by scalars) is defined on $\mathbb{R} \times \mathcal{M}$ satisfying:

(i) $\gamma * (A + B) = \gamma * A + \gamma * B$,

(ii) $\alpha * (\beta * C) = (\alpha\beta) * C$,

(iii) $1 * A = A$,

(iv) $(\alpha + \beta) * C = \alpha * C + \beta * C$, if $\alpha\beta \geqslant 0$.

The algebraic system $(\mathcal{M}, +, \mathbb{R}, *)$ is called a *quasilinear space over* $\mathbb{R}$.

Every quasilinear space $(\mathcal{M}, +, \mathbb{R}, *)$ can be embedded into a group $(\mathcal{D}(\mathcal{M}), +)$. Multiplication by scalars "$*$" is naturally extended from $\mathbb{R} \times \mathcal{M}$ to $\mathbb{R} \times \mathcal{D}(\mathcal{M})$ by means of

$$\gamma * (A, B) = (\gamma * A, \gamma * B), \quad A, B \in \mathcal{M},\ \gamma \in \mathbb{R}. \tag{1}$$

In the sequel we shall call quasilinear spaces of group structure, such as $\mathcal{D}(\mathcal{M})$, quasivector spaces, and denote their elements by lower case roman letters, e.g., $a = (A_1, A_2)$, $A_1, A_2 \in \mathcal{M}$. Quasivector spaces are defined as follows [5]:

Definition 2. A *quasivector space* (*over* $\mathbb{R}$), denoted $(\mathcal{Q}, +, \mathbb{R}, *)$, is an Abelian group $(\mathcal{Q}, +)$ with a mapping (multiplication by scalars) "$*$": $\mathbb{R} \times \mathcal{Q} \longrightarrow \mathcal{Q}$, such that for $a, b, c \in \mathcal{Q}$, $\alpha, \beta, \gamma \in \mathbb{R}$: $\gamma * (a + b) = \gamma * a + \gamma * b$, $\alpha * (\beta * c) = (\alpha\beta) * c$, $1 * a = a$, $(\alpha + \beta) * c = \alpha * c + \beta * c$, if $\alpha\beta \geqslant 0$.

Proposition 1 [4]. Let $(\mathcal{M}, +, \mathbb{R}, *)$ be a quasilinear space over $\mathbb{R}$, and $(\mathcal{Q}, +)$, $\mathcal{Q} = \mathcal{D}(\mathcal{M})$, be the induced Abelian group. Let $* : \mathbb{R} \times \mathcal{Q} \longrightarrow \mathcal{Q}$ be multiplication by scalars defined by (1). Then $(\mathcal{Q}, +, \mathbb{R}, *)$ is a quasivector space over $\mathbb{R}$.

Let a be an element of a quasivector space $(\mathcal{Q}, +, \mathbb{R}, *)$, $a \in \mathcal{Q}$. The operator $\neg a = (-1) * a$ is called *negation*; in the literature it is usually denoted $-a = (-1) * a$. We write $a \neg b = a + (\neg b)$; note that $a \neg a = 0$ may not generally hold. From $\mathrm{opp}(a) + a = 0$ we obtain $\neg\mathrm{opp}(a) \neg a = 0$, that is $\neg\mathrm{opp}(a) = \mathrm{opp}(\neg a)$. We shall use the notation $a_- = \neg\mathrm{opp}(a) = \mathrm{opp}(\neg a)$; the latter operator is called *dualization* or *conjugation*. The relations $\neg\mathrm{opp}(a) = \mathrm{opp}(\neg a) = a_-$ imply $\mathrm{opp}(a) = \neg(a_-) = (\neg a)_-$, shortly $\mathrm{opp}(a) = \neg a_-$. Thus, the symbolic notation $\neg a_-$ can be used instead of $\mathrm{opp}(a)$, and, for $a \in \mathcal{Q}$ we can write $a \neg a_- = 0$, respectively $\neg a_- + a = 0$. Many vector space concepts, such as subspace, sum and direct sum, are trivially extended to quasivector spaces. Rules for calculation in quasivector spaces are summarized in [5].

Example 1. For any integer $k \geqslant 1$ the set $\mathbb{R}^k$ of all k-tuples $(\alpha_1, \alpha_2, \dots, \alpha_k)$, $\alpha_i \in \mathbb{R}$, with $(\alpha_1, \alpha_2, \dots, \alpha_k) = (\beta_1, \beta_2, \dots, \beta_k)$ whenever $\alpha_1 = \beta_1, \alpha_2 = \beta_2, \dots, \alpha_k = \beta_k$, forms a quasivector space over $\mathbb{R}$ under the operations

$$\begin{aligned}(\alpha_1, \alpha_2, \dots, \alpha_k) + (\beta_1, \beta_2, \dots, \beta_k) &= (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots, \alpha_k + \beta_k),\\ \gamma * (\alpha_1, \alpha_2, \dots, \alpha_k) &= \big(|\gamma|\alpha_1, |\gamma|\alpha_2, \dots, |\gamma|\alpha_k\big), \quad \gamma \in \mathbb{R}.\end{aligned}$$

This quasivector space is denoted by $\mathbb{S}^k = (\mathbb{R}^k, +, \mathbb{R}, *)$. Negation in $\mathbb{S}^k$ is the same as identity while the opposite operator is the same as conjugation:

$$\mathrm{opp}(\alpha_1, \alpha_2, \dots, \alpha_k) = (\alpha_1, \alpha_2, \dots, \alpha_k)_- = (-\alpha_1, -\alpha_2, \dots, -\alpha_k). \tag{2}$$

The direct sum $\mathbb{V}^l \oplus \mathbb{S}^k$ of the l-dimensional vector space $\mathbb{V}^l = (\mathbb{R}^l, +, \mathbb{R}, \cdot)$ and the quasivector space $\mathbb{S}^k = (\mathbb{R}^k, +, \mathbb{R}, *)$ is a quasivector space.

Example 2. The system $(\mathcal{K}, +)$ of all convex bodies [8] in a real m-dimensional Euclidean vector space $\mathbb{E}^m$ with addition: $A + B = \{a + b \mid a \in A, b \in B\}$, $A, B \in \mathcal{K}$, is an Abelian monoid with cancellation law having as a neutral element the origin "0" of $\mathbb{E}^m$. The system $(\mathcal{K}, +, \mathbb{R}, *)$, where "$*$" is multiplication by real scalars defined by: $\gamma * A = \{\gamma a \mid a \in A\}$, is a quasilinear space (of monoid structure), that is the following four relations are satisfied:

(i) $\gamma * (A + B) = \gamma * A + \gamma * B$,

(ii) $\alpha * (\beta * C) = (\alpha\beta) * C$,

(iii) $1 * A = A$,

(iv) $(\alpha + \beta) * C = \alpha * C + \beta * C$, if $\alpha\beta \geqslant 0$ [5].

The monoid $(\mathcal{K}, +)$ induces a group of generalized convex bodies $(\mathcal{D}(\mathcal{K}), +)$, cf. [1]. In [4] we investigate the space $(\mathcal{D}(\mathcal{K}), +, \mathbb{R}, *)$, where "$*$" is defined by (1).

Definition 3. Q is a quasivector space. An element $a \in \mathcal{Q}$ with $a \neg a = 0$ is called *linear*. An element $a \in \mathcal{Q}$ with $\neg a = a$ is called *centred* or *origin symmetric*.

Proposition 2. Assume that Q is a quasivector space. The subsets of linear and centred elements $\mathcal{Q}' = \{a \in \mathcal{Q} \mid a \neg a = 0\}$, respectively $\mathcal{Q}'' = \{a \in \mathcal{Q} \mid a = \neg a\}$ form subspaces of $\mathcal{Q}$. The subspace $\mathcal{Q}'$ is a vector space.

Definition 4. Assume that Q is a quasivector space. The space $\mathcal{Q}' = \{a \in \mathcal{Q} \mid a \neg a = 0\}$ is called the *linear subspace* of $\mathcal{Q}$ and the space $\mathcal{Q}'' = \{a \in \mathcal{Q} \mid a = \neg a\}$ is called the *symmetric subspace* or *centred subspace* of $\mathcal{Q}$.

Theorem 1 [5]. For every quasivector space $\mathcal{Q}$ we have $\mathcal{Q} = \mathcal{Q}' \oplus \mathcal{Q}''$. More specifically, for every $x \in \mathcal{Q}$ we have $x = x' + x'' = (x'; x'')$ with unique $x' = (1/2) * (x + x_-) \in \mathcal{Q}'$, and $x'' = (1/2) * (x \neg x) \in \mathcal{Q}''$.

Symmetric quasivector spaces and their relation to vector spaces. Let $(\mathcal{Q}, +, \mathbb{R}, *)$ be a (symmetric) quasivector space over $\mathbb{R}$. For $\gamma \in \mathbb{R}$ denote $\sigma(\gamma) = \{+, \text{if } \gamma \geqslant 0; -, \text{if } \gamma < 0\}$ and $c_+ = c$. Consider the operation "$\cdot$": $\mathbb{R} \times \mathcal{Q} \to \mathcal{Q}$ defined by

$$\alpha \cdot c = \alpha * c_{\sigma(\alpha)} = \begin{cases} \alpha * c, & \text{if } \alpha \geqslant 0, \\ \alpha * c_-, & \text{if } \alpha < 0. \end{cases} \tag{3}$$

Theorem 2 [4,5]. Let $(\mathcal{Q}, +, \mathbb{R}, *)$ be a symmetric quasivector space over $\mathbb{R}$. Then $(\mathcal{Q}, +, \mathbb{R}, \cdot)$, with "$\cdot$" defined by (3), is a vector space over $\mathbb{R}$.

Note that for a centred, the element $(-1) \cdot a = (-1) * a_- = a_-$ is the opposite to a, that is $a + (-1) \cdot a = 0$, respectively $a + a_- = 0$.

Linear combinations. Assume that $(\mathcal{S}, +, \mathbb{R}, *)$ is a *symmetric* quasivector space and $(\mathcal{S}, +, \mathbb{R}, \cdot)$ is the associated vector space according to theorem 2. We may transfer vector space concepts from $(\mathcal{S}, +, \mathbb{R}, \cdot)$, such as linear combination, linear dependence, basis etc., to the original symmetric quasivector space $(\mathcal{S}, +, \mathbb{R}, *)$. For example, let $c^{(1)}, c^{(2)}, \cdots, c^{(k)}$ be finitely many elements of $\mathcal{S}$. The familiar linear combination $f = \sum_{i=1}^{k} \alpha_i \cdot c^{(i)} = \alpha_1 \cdot c^{(1)} + \alpha_2 \cdot c^{(2)} + \cdots + \alpha_k \cdot c^{(k)}$, $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}$, in the induced vector space $(\mathcal{S}, +, \mathbb{R}, \cdot)$, can be rewritten using (3) as

$$f = \alpha_1 * c^{(1)}_{\sigma(\alpha_1)} + \alpha_2 * c^{(2)}_{\sigma(\alpha_2)} + \cdots + \alpha_k * c^{(k)}_{\sigma(\alpha_k)}. \tag{4}$$

Thus (4) is a *linear combination* of $c^{(1)}, c^{(2)}, \ldots, c^{(k)} \in \mathcal{S}$ in the symmetric quasivector space $(\mathcal{S}, +, \mathbb{R}, *)$. Similarly, the concepts of spanned subspace, linear (in)dependency, linear mapping, basis, dimension, etc. are defined, and the theory of vector spaces can be reformulated in $(\mathcal{S}, +, \mathbb{R}, *)$ [4,5].

Theorem 3 [5]. Any symmetric quasivector space over $\mathbb{R}$, with a basis of k elements, is isomorphic to $\mathbb{S}^k = (\mathbb{R}^k, +, \mathbb{R}, *)$.

3. Computation with zonotopes in the plane

Our next aim is to concentrate to examples 1 and 2 having in mind a particular class of convex bodies and the basic results of the above outlined theory. In what follows we restrict ourselves to consideration of convex bodies in the Euclidean plane $\mathbb{E}^2$, see example 2.

Let $(\mathcal{K}, +, \mathbb{R}, *)$ be a system of convex bodies in $\mathbb{E}^2$, partially ordered by the inclusion relation $\subseteq$. Denote by $\langle\cdot,\cdot\rangle$ the inner product and by $|\cdot|$ – the norm in $\mathbb{E}^2$, and let $\mathcal{U}$ be the unit ball, and $\mathcal{S}$ – the unit sphere in $\mathbb{E}^2$. The norm in $\mathcal{K}$ is $|A| = \max\{|a|\colon a \in A\}$. The support function of $A \in \mathcal{K}$ is defined by $h(A, u) = \max\{\langle a, u\rangle\colon a \in A\}$ [8,9]. The excess from A to B is

$$\operatorname{excess}(A, B) = \inf\{\alpha > 0\colon A \subseteq B + \alpha * \mathcal{U}\},$$

and the Hausdorff distance is defined in $\mathcal{K}$ by

$$\operatorname{haus}(A, B) = \max\bigl\{\operatorname{excess}(A, B), \operatorname{excess}(B, A)\bigr\}.$$

The following relation exists between the Hausdorff distance in $\mathcal{K}$ and the uniform distance for the support functions

$$\operatorname{haus}(A, B) = \max\bigl\{\bigl|h(A, e) - h(B, e)\bigr|\colon e \in S\bigr\}, \quad A, B \in \mathcal{K}. \tag{5}$$

A centrally symmetric convex body with center at the origin is called *centred convex body* (cf. [8, p. 383]). Every unit vector in $\mathbb{E}^2$: $e = (\cos\varphi, \sin\varphi) \in \mathbb{E}^2$, $\varphi \in [0, \pi)$, defines a centred segment $\tilde{e} \in \mathcal{K}$ with endpoints $-e$ and e:

$$\tilde{e} = \operatorname{conv}\{-e, e\} = \bigl\{\lambda e \mid \lambda \in [-1, 1]\bigr\},$$

where conv means the convex hull, see [8]. If v is a vector, then $\tilde{v}$ or $v\tilde{}$ means the corresponding centred segment induced by the vector: $\tilde{v} = \operatorname{conv}\{-v, v\}$. Note that $v \in \mathbb{R}^2$, whereas $\tilde{v}$ is a convex body, that is $\tilde{v} \in \mathcal{K}$.

For $\rho \in \mathbb{R}$ denote $s = \rho e$. Multiplication of a unit centred segment $\tilde{e}$ by a scalar $\rho \in \mathbb{R}$ is:

$$\tilde{s} = \rho * \tilde{e} = (\rho e)\tilde{} = \operatorname{conv}\{-s, s\} = \bigl\{\lambda\rho e \mid \lambda \in [-1, 1]\bigr\}.$$

Multiplication of a centred (not necessarily unit) segment $(\rho e)\tilde{}$ by a scalar $\gamma \in \mathbb{R}$ satisfies:

$$\gamma * (\rho e)\tilde{} = \bigl((\gamma\rho)e\bigr)\tilde{} = (\gamma\rho) * \tilde{e}.$$

Note that $-1 * \tilde{s} = \tilde{s}$; more generally, $-\rho * \tilde{s} = \rho * \tilde{s}$ (for comparison, of course, $\rho s \neq -\rho s$).

Minkowski sum of colinear segments is $(\rho_1 e)\tilde{} + (\rho_2 e)\tilde{} = ((\rho_1 + \rho_2)e)\tilde{}$. To present Minkowski sum of two noncolinear segments, let us assume that $0 \leqslant \varphi_1 < \varphi_2 < \pi$ and denote $e^{(i)} = (\cos\varphi_i, \sin\varphi_i)$, $i = 1, 2$. The points

$$s^{(1)} = \rho_1 e^{(1)} = (\rho_1\cos\varphi_1, \rho_1\sin\varphi_1),$$
$$s^{(2)} = \rho_2 e^{(2)} = (\rho_2\cos\varphi_2, \rho_2\sin\varphi_2),$$

where $\rho_1, \rho_2 \in \mathbb{R}$, define two noncolinear centred segments $\tilde{s}^{(1)}, \tilde{s}^{(2)} \in \mathcal{K}$. The Minkowski sum $\tilde{s}^{(1)} + \tilde{s}^{(2)}$ is a centred parallelogram p with vertices $\{t^{(1)}, t^{(2)}, -t^{(1)}, -t^{(2)}\}$, where $t^{(1)} = s^{(1)} + s^{(2)}$, $t^{(2)} = -s^{(1)} + s^{(2)}$. The perimeter of $p = \operatorname{conv}\{t^{(1)}, t^{(2)}, -t^{(1)}, -t^{(2)}\}$ is $4(\rho_1 + \rho_2)$ and the area of p is $4\rho_1\rho_2 \sin(\varphi_2 - \varphi_1)$.

In the sequel we shall assume that we are given a mesh of k numbers φ_i in the interval $[0, \pi)$, such that

$$0 \leqslant \varphi_1 < \varphi_2 < \cdots < \varphi_k < \pi. \tag{6}$$

A system of numbers of the form (6) will be called *regular*. Every φ_i defines a unit vector $e^{(i)} = (\cos\varphi_i, \sin\varphi_i)$, and respectively a unit segment: $\tilde{e}^{(i)} = \operatorname{conv}\{-e^{(i)}, e^{(i)}\}$. The systems of vectors, respectively segments:

$$e^{(1)}, e^{(2)}, \ldots, e^{(k)} \in \mathbb{E}^2, \qquad \tilde{e}^{(1)}, \tilde{e}^{(2)}, \ldots, \tilde{e}^{(k)} \in \mathcal{K}, \tag{7}$$

will be also called *regular*; they are cyclically anticlockwise ordered.

In particular, we may assume an uniform system of points $\varphi_i = \pi(i-1)/k$, $i = 1, \ldots, k$; the respective systems $\{e^{(i)}\}_{i=1}^k$, $\{\tilde{e}^{(i)}\}_{i=1}^k$ will be also called *uniform*.

For $\alpha_i \geqslant 0$ the vectors $s^{(i)} = \alpha_i e^{(i)} = (\alpha_i \cos\varphi_i, \alpha_i \sin\varphi_i)$ induce centred segments $\tilde{s}^{(i)} = \alpha_i * \tilde{e}^{(i)} = (\alpha_i e^{(i)})\tilde{}$, $i = 1, \ldots, k$. The *positive combination* of segments $\tilde{e}^{(i)}$

$$z = \sum_{i=1}^k \tilde{s}^{(i)} = \sum_{i=1}^k \alpha_i * \tilde{e}^{(i)}, \quad \alpha_i \geqslant 0, \tag{8}$$

is a centred *zonotope* with $2k$ vertices: $t^{(1)}, t^{(2)}, \ldots, t^{(k)}, -t^{(1)}, -t^{(2)}, \ldots, -t^{(k)}$ [8], that is $z = \operatorname{conv}\{t^{(1)}, t^{(2)}, \ldots, t^{(k)}, -t^{(1)}, -t^{(2)}, \ldots, -t^{(k)}\}$, where

$$\begin{aligned}
t^{(1)} &= \alpha_1 e^{(1)} + \alpha_2 e^{(2)} + \cdots + \alpha_{k-1} e^{(k-1)} + \alpha_k e^{(k)},\\
t^{(2)} &= -\alpha_1 e^{(1)} + \alpha_2 e^{(2)} + \cdots + \alpha_{k-1} e^{(k-1)} + \alpha_k e^{(k)},\\
&\vdots\\
t^{(i)} &= -\alpha_1 e^{(1)} - \cdots - \alpha_{i-1} e^{(i-1)} + \alpha_i e^{(i)} + \cdots + \alpha_k e^{(k)},\\
&\vdots\\
t^{(k)} &= -\alpha_1 e^{(1)} - \alpha_2 e^{(2)} + \cdots - \alpha_{k-1} e^{(k-1)} + \alpha_k e^{(k)}.
\end{aligned} \tag{9}$$

The vertices $t^{(1)}, t^{(2)}, \ldots, t^{(k)}$ given by (9) are lying in ciclic order anticlockwise in a half-plane between the vectors $t^{(1)}$ and $t^{(k)} = -t^{(1)} + 2\alpha_k e^{(k)}$. The perimeter of the zonotope $z = \sum_{i=1}^k \alpha_i * \tilde{e}^{(i)}$ is $4(\alpha_1 + \alpha_2 + \cdots + \alpha_k)$ and the area of z is $4\sum_{i,j=1, j>i}^k \alpha_j \alpha_i \sin(\varphi_j - \varphi_i)$.

Two centred zonotopes $b = \sum_{i=1}^k \beta_i * \tilde{e}^{(i)}$, $c = \sum_{i=1}^k \gamma_i * \tilde{e}^{(i)}$, $\beta_i \geqslant 0$, $\gamma_i \geqslant 0$, over same regular system $\{\tilde{e}^{(i)}\}_{i=1}^k$, are added by $b + c = \sum_{i=1}^k \beta_i * \tilde{e}^{(i)} + \sum_{i=1}^k \gamma_i * \tilde{e}^{(i)} = \sum_{i=1}^k (\beta_i + \gamma_i) * \tilde{e}^{(i)}$. Thus given a fixed regular system of centred unit segments $\{\tilde{e}^{(i)}\}_{i=1}^k$, the set of all zonotopes of the form (8): $\sum_{i=1}^k \alpha_i * \tilde{e}^{(i)}$, $\alpha_i \geqslant 0$, is closed under Minkowski addition and multiplication by scalars and forms a quasilinear space (of monoid structure).

If we add two zonotopes $b = \sum_{i=1}^{k} \beta_i * \tilde{u}^{(i)}$, $c = \sum_{i=1}^{l} \gamma_i * \tilde{v}^{(i)}$, where $\{\tilde{u}^{(i)}\}_{i=1}^{k}$, $\{\tilde{v}^{(i)}\}_{i=1}^{l}$, are two distinct systems of centred unit segments, then we see from the expression for the sum: $b + c = \sum_{i=1}^{k} \beta_i * \tilde{u}^{(i)} + \sum_{i=1}^{l} \gamma_i * \tilde{v}^{(i)}$ that the vertices of $b + c$ can be restored using (9); clearly the number of vertices of $b + c$ equals (generally) the sum $k + l$ of the numbers of vertices of b and c, respectively. If we want to use a fixed presentation of the zonotopes of the form (8), then we need to present (approximately) all zonotopes using one and the same system of centred unit segments. We thus arrive to an approximation problem to be considered next.

4. An approximation problem for zonotopes

Assume that a regular basic system of $k \geqslant 2$ mesh points (angles) $\{\varphi_i\}_{i=1}^{k}$, is given generating a basic system of centred unit segments

$$\tilde{e}^{(1)}, \tilde{e}^{(2)}, \ldots, \tilde{e}^{(k)}, \quad e^{(i)} = (\cos\varphi_i, \sin\varphi_i). \tag{10}$$

Denote by Z_p^k the set of all *positive combinations* of centred segments (10) in the form (8): $z = \sum_{i=1}^{k} \varepsilon_i * \tilde{e}^{(i)}$, $\varepsilon_i \geqslant 0$. (The letter p in Z_p^k stands for “positive” or “proper”.) Assume that a (new) system of $m \geqslant 1$ mesh points $\{\psi^{(i)}\}_{i=1}^{m}$ is given such that $0 \leqslant \varphi_1 \leqslant \psi_1 < \psi_2 < \cdots < \psi_m \leqslant \varphi_k < \pi$. The system $\{\psi^{(i)}\}_{i=1}^{m}$ generates a regular system $\{\tilde{p}^{(i)}\}_{i=1}^{m}$ of unit centred segments, distinct from the given system $\{\tilde{e}^{(i)}\}_{i=1}^{k}$. Given the numbers $\rho_i \geqslant 0, i = 1, \ldots, m$, we want to approximate the zonotope $w = \sum_{i=1}^{m} \rho_i * \tilde{p}^{(i)}$ by means of zonotopes from the class Z_p^k, so that $w \subseteq z$.

Inclusion. Every vector $p^{(i)} = (\cos\psi_i, \sin\psi_i)$ can be presented as

$$p^{(i)} = \delta_{i1} e^{(j)} + \delta_{i2} e^{(j+1)}, \quad j = j(i),\ i = 1, \ldots, m, \tag{11}$$

where $e^{(j)}, e^{(j+1)}$ are the nearest basic unit vectors enclosing $p^{(i)}$ with $\varphi_j \leqslant \psi_i \leqslant \varphi_{j+1}$ and δ_{i1}, δ_{i2} are some nonnegative coefficients. Denote $d^{(i)} = \delta_{i1} * \tilde{e}^{(j)} + \delta_{i2} * \tilde{e}^{(j+1)}$, $i = 1, \ldots, m$. The zonotope $d^{(i)}$ is a centred parallelogram, which contains the segment $\tilde{p}^{(i)}$. Using that the zonotope $d^{(i)}$ contains the segment $\tilde{p}^{(i)}$, $\tilde{p}^{(i)} \subseteq d^{(i)}$, $i = 1, \ldots, m$, we obtain

$$\begin{aligned} w &= \sum_{i=1}^{m} \rho_i * \tilde{p}^{(i)} \subseteq \sum_{i=1}^{m} \rho_i * d^{(i)} = \sum_{i=1}^{m} \rho_i * \left(\delta_{i1} * \tilde{e}^{(j)} + \delta_{i2} * \tilde{e}^{(j+1)}\right) \\ &= \sum_{i=1}^{m} \left(\rho_i \delta_{i1} * \tilde{e}^{(j)} + \rho_i \delta_{i2} * \tilde{e}^{(j+1)}\right) = \sum_{i=1}^{k} \varepsilon_i * \tilde{e}^{(i)} = z \end{aligned} \tag{12}$$

with some $\varepsilon_i \geqslant 0$ that can be effectively computed. It follows from (12) that the zonotope z satisfies the inclusion $w \subseteq z$. To proof that z approximates the zonotope w we shall compute the difference in the areas of the two zonotopes and shall show that this difference tends to zero with the refinement of the basic mesh.

B. Approximation. In order to discuss the approximation part we need the following lemmas.

Lemma 1. The area of the centred zonotope $d^{(i)} = \delta_{i1} * \tilde{e}^{(j)} + \delta_{i2} * \tilde{e}^{(j+1)}$, approximating the segment $\tilde{p}^{(i)}$, does not exceed $2\tan((\varphi^{(j+1)} - \varphi^{(j)})/2)$, respectively for the uniform case: $\tan(\pi/(2k))$.

Lemma 2. For a centred zonotope z, the area $S(z)$ of z satisfies: $S(\rho * z) = \rho^2 S(z)$, $\rho \geqslant 0$.

Lemma 3. Assume that p and q are two different vectors from the system $\{p^{(i)}\}_{i=1}^{m}$. Assume that

$$p = \gamma_1 e^{(i)} + \gamma_2 e^{(i+1)},$$
$$q = \delta_1 e^{(j)} + \delta_2 e^{(j+1)},$$

where $e^{(i)}$, $e^{(i+1)}$ are the nearest unit vectors from (10), enclosing p and $e^{(j)}$, $e^{(j+1)}$ are the nearest unit vectors from (10), enclosing q, and $\gamma_1, \gamma_2, \delta_1, \delta_2$ are some nonnegative coefficients. Denote

$$u = \gamma_1 * \tilde{e}^{(i)} + \gamma_2 * \tilde{e}^{(i+1)},$$
$$v = \delta_1 * \tilde{e}^{(j)} + \delta_2 * \tilde{e}^{(j+1)}.$$

Then the following relation for the area S of the zonotopes u, v, $u + v$ and $\tilde{p} + \tilde{q}$ takes place:

$$S(u + v) = S(u) + S(v) + S(\tilde{p} + \tilde{q}). \tag{13}$$

More generally we have

$$S(\alpha * u + \beta * v) = S(\alpha * u) + S(\beta * v) + S(\alpha * \tilde{p} + \beta * \tilde{q}). \tag{14}$$

where α, β are nonnegative numbers.

Using the construction of z one can obtain better approximation by subsequent refinement of the basic mesh system. By a mesh refinement we mean the construction of an infinite sequence of regular basic mesh systems such that the distance between the mesh points (angles) tends to zero. The simplest such mesh system is the uniform one with $\varphi_i^{(k)} = \pi(i - 1)/k$, $i = 1, \ldots, k$, where k goes to infinity (say, as $k = 2^t$).

It can be noticed from (12) that the approximating zonotope $z_k = \sum_{i=1}^{k} \varepsilon_i * \tilde{e}^{(i)}$ has not more that $2m$ summands, that is $k \leqslant 2m$, where m is a fixed integer. Thus the approximating zonotope z_k is a finite sum of (at most) m summands $z = \sum_{i=1}^{m} \rho_i * d^{(i)}$ and each of these summands $\rho_i * d^{(i)}$ tends to the corresponding summand $\rho_i * \tilde{p}^{(i)}$ in the sum $w = \sum_{i=1}^{m} \rho_i * \tilde{p}^{(i)}$. Hence the sequence of zonotopes z_k tends in Hausdorff sense to the given zonotope w. We thus arrived the following:

Theorem 4. Given a centred zonotope w, there exists a sequence of zonotopes z_k tending to w in Hausdorff sense such that $w \subseteq z_k$. For the case of an uniform mesh $\tilde{e}^{(1)}, \ldots, \tilde{e}^{(k)}$ we obtain $S(z_k) - S(w) = \mathrm{O}(1/k)$.

Proof. According to lemma 1 the area of the zonotope $u = \gamma_1 * \tilde{e}^{(i)} + \gamma_2 * \tilde{e}^{(i+1)}$ does not exceed $2\tan(\varphi_{i+1} - \varphi_i)/2$, which can be used to compute the area of z_k, and compare it to the area of $w \subseteq z$. Using lemmas 2, 3 we obtain

$$S(z_k) - S(w) = 2\sum_{i=1}^{m} \rho_i^2 \tan\frac{\varphi_{i+1} - \varphi_i}{2}.$$

In the case of uniform mesh $\varphi_{i+1} - \varphi_i = \pi/k$ we obtain

$$S(z_k) - S(w) = 2\tan\frac{\pi}{2k}\sum_{i=1}^{m} \rho_i^2,$$

which proves the proposition. $\square$

From (12) one can compute the vertices of the zonotope z by means of (9) or the supporting halfplanes.

It can be shown that the above algorithm produces an optimal approximation as regard to the Hausdorff metric. This follows from the fact that every single segment (11): $p^{(i)} = \delta_{i1}e^{(j)} + \delta_{i2}e^{(j+1)}$ has been optimally approximated by the parallelogram $d^{(i)} = \delta_{i1} * \tilde{e}^{(j)} + \delta_{i2} * \tilde{e}^{(j+1)}$. Note that the area of the zonotope $\varepsilon_{i1} * \tilde{e}^{(j)} + \varepsilon_{i2} * \tilde{e}^{(j+1)}$ is $4\varepsilon_{i1}\varepsilon_{i2}\sin(\varphi_{j+1} - \varphi_j)$.

Relations (11), (12) suggest the formulation of an algorithm leading to the construction of the zonotope z giving an outer approximation of w. An algorithm using a regular uniform mesh of centred segments in $\mathbb{E}^2$ has been realized in the computer algebra systems Matlab and Mathematica.

The method of support functions

Let us compare our method with a corresponding method based on support functions. The support function of a set $A \in \mathbb{E}^2$, namely $h(A; u) = \max_{x\in A}\langle x, u\rangle$ is well defined by its values on the unit circle $\mathcal{S}$, that is for $u = (\cos\theta, \sin\theta) \in \mathcal{S}$. Let us calculate $h(A; u)$ for a zonotope A. The simplest zonotope is the centred unit segment $\tilde{e}$ with $e(\varphi) = (\cos\varphi, \sin\varphi)$. We have $h(\tilde{e}; u) = h(\tilde{e}(\varphi); u) = |\cos\varphi\cos\theta + \sin\varphi\sin\theta| = |\cos(\theta - \varphi)|$.

Using this expression we can write down the support function of the zonotope $z = \sum_{i=1}^{k} \varepsilon_i * \tilde{e}^{(i)}$, where $e^{(i)} = (\cos\varphi_i, \sin\varphi_i)$, $i = 1, \ldots, k$, $0 \leqslant \varphi_1 \leqslant \cdots \leqslant \varphi_m < \pi$. We have $h(z, \theta) = \max_{x\in\tilde{e}}\langle x, u\rangle = \sum_{i=1}^{k} \varepsilon_i |\langle e^{(i)}, \cdot\rangle| = \sum_{i=1}^{k} \varepsilon_i h(\tilde{e}^{(i)}; \theta) = \sum_{i=1}^{k} \varepsilon_i |\cos(\theta - \varphi_i)|$.

The above approximation problem can be stated in terms of support functions as follows. Given some ψ_i, $i = 1, \dots, m$, $0 \leqslant \psi_1 \leqslant \cdots \leqslant \psi_m < \pi$, we want to approximate from above in the interval $[0, \pi]$ the function $w(\theta) = \sum_{i=1}^{m} \varepsilon_i |\cos(\theta - \psi_i)|$ by means of a function of the form $z_k(\theta) = \sum_{i=1}^{k} \varepsilon_i |\cos(\theta - \varphi_i)|$, that is we need to find the ε's in z_k so that $z_k \geqslant w$ and z_k approximates w. In view of (5) the latter means that $\max_{\theta \in [0,\pi)} |w(\theta) - z_k(\theta)|$ has to be minimized (w.r.t. the ε's).

Generalized zonotopes

As discussed in section 2 we can introduce new *improper* elements in the set of centred zonotopes on $\mathbb{E}^2$ of the form (8), for a fixed k, so that this set becomes a quasivector space (under Minkowski addition and multiplication by scalar). The obvious way to do this is by extending the positive combination (8) in the spirit of (4), that is, to fix a regular system of $k \geqslant 2$ centred segments (10) and to consider the set $Z^k = \mathcal{D}(Z_p^k)$ of all *linear combinations* of centred segments in the form:

$$z = \sum_{i=1}^{k} \alpha_i \cdot \tilde{e}^{(i)} = \sum_{i=1}^{k} \alpha_i * \tilde{e}^{(i)}_{\sigma(\alpha_i)}, \quad (\alpha_1, \alpha_2, \dots, \alpha_k) \in \mathbb{R}^k. \tag{15}$$

The elements of Z^k are called *generalized zonotopes*. Now, according to theorem 3, the space $\mathcal{D}(Z^k)$ is a symmetric quasivector space isomorphic to the space $\mathbb{S}^k = (\mathbb{R}^k, +, \mathbb{R}, *)$ from example 1. According to theorem 2 the latter space is equivalent to the vector space $\mathbb{V}^k$. We thus see that the vector space $\mathbb{V}^k$ with the familiar addition of vectors and multiplication by scalars is equivalent to the symmetric quasivector space of generalized centred zonotopes of the form (15). In this manner computations with centred zonotopes are reduced to computations in a familiar vector space.

The opposite element to $\tilde{e}$ is the improper segment $\tilde{e}_-$, such that $\tilde{e} + \tilde{e}_- = 0$. The element $\tilde{e}_-$ can be interpreted (visualized) as the set of all points on the real line $\mathbb{R}$ supporting the segment (that is, passing through the vectors e, $-e$) without the points of the segment $\tilde{e}$, i.e. $\tilde{e}_- = \mathbb{R} \setminus \tilde{e}$. Another interpretation (related to concepts like normals) uses directions: $\tilde{e}$ has a direction from $-e$ to e, whereas $\tilde{e}_-$ has an opposite direction (from e to $-e$).

According to theorems 2, 3 the space of generalized centred segments over the real line $\mathbb{R}$ is equivalent to a vector space with a *linear multiplication* "$\cdot$" defined by $\alpha\tilde{s} = \alpha \cdot \tilde{s} = \alpha * \tilde{s}_{\sigma(\alpha)}$, $\alpha \in \mathbb{R}$. In particular, for the opposite segment of $\tilde{s}$ we have $(-1) \cdot \tilde{s} = (-1) * \tilde{s}_- = \tilde{s}_-$. Using the usual notation for opposite in a linear space we may write $-\tilde{s} = (-1) \cdot \tilde{s}$ for $\tilde{s}_-$. Then we may write expressions like $\tilde{s} - \tilde{s} = 0$, $\alpha\tilde{s} = \alpha \cdot \tilde{s}$, etc., but we should be careful with the meaning of such expressions, e.g., $\tilde{s} - \tilde{s} = \tilde{s} + \tilde{s}_-$, $\alpha\tilde{s} = \alpha * \tilde{s}_{\sigma(\alpha)}$, respectively. Furthermore, one can assume that an improper segment is generated by a vector $(\cos\varphi, \sin\varphi)$ with $\varphi \in [\pi, 2\pi)$. Then we could write, e.g., $-\tilde{e} = \tilde{e}_- = (-e)\tilde{}$.

A (generalized) zonotope (15) can be identified with the vector $(\alpha_1, \alpha_2, \dots, \alpha_k) \in \mathbb{R}^k$. Every term $\alpha_i \cdot \tilde{e}^{(i)}$ in (15) having a negative coefficient $\alpha_i < 0$ is improper, in this case: $\alpha_i \cdot \tilde{e}^{(i)} = \alpha_i * \tilde{e}^{(i)}_{\sigma(\alpha_i)} = \alpha_i * \tilde{e}^{(i)}_{-}$. A zonotope is improper if at least one α_i is negative. Recall that the set of all points x satisfying the inequalities:

$$\langle x - t^{(i)}, s^{(i)*} \rangle \leqslant 0, \qquad \langle x + t^{(i)}, -s^{(i)*} \rangle \leqslant 0, \tag{16}$$

for some fixed $i = 1, \ldots, k$ is a *strip* in $\mathbb{E}^2$. The zonotope z is an intersection of all k strips (16) with $i = 1, \ldots, k$. A generalized zonotope (15) can be visualized as an intersection of strips that can be proper or improper. An improper strip corresponding to a negative α_i, respectively to an improper segment $\alpha_i \tilde{e}^{(i)}$ can be visualized as the complement to the respective (proper) strip in $\mathbb{E}^2$; note that the direction of the normals $\tilde{s}^{(i)} = \alpha_i \tilde{e}^{(i)}$ in (16) depends on the sign of α_i. Another possible way to vizualize generalized zonotopes is to endow the boundary of the corresponding proper zonotope with arrows pointing outward or inward depending on the sign of the corresponding α_i. A similar visualization has been suggested in [1].

By now we have considered only centred (generalized) zonotopes. The consideration of zonotopes that are not centred in the origin presents no problem according to theorem 1, as they are sums of vectors and centred generalized zonotopes, cf. example 1.

5. Concluding remarks

It has been widely recognized that zonotopes are a suitable tool for bounding regions of uncertainty. However, using zonotopes generated by arbitrary segments makes computations extremely complex due to large (potentially infinite) number of parameters. Therefore, it is desirable to consider zonotopes from a finite-parametric family, with a fixed number of parameters; such a natural family of regular basic vectors has been used in the paper. Fixing a simple family of zonotopes, then the natural problem of approximating all zonotopes by means of zonotopes from this family arrises. We then suggest an algorithm producing approximate enclosure and show that this enclosure is optimal with respect to the Hausdorff distance.

In the present study of the presentation and algebraic computation with zonotopes we have been guided by the theory of quasivector spaces. As every quasivector space is a direct sum of a linear subspace and a symmetric quasivector subspace we concentrate on the space of centrally symmetric zonotopes centred at the origin (centred zonotopes) which can be represented as Minkowskian sums of centred segments. Spaces of such zonotopes can be naturally embedded in symmetric quasivector spaces with group structure. We show how this can be done and give an idea how improper elements can be represented and visualized. Our approach is alternative to the approach of support functions, extensively used in the literature on convex bodies. The space of extended (generalized, differences of) support functions is a quasivector space, and we can calculate with generalized convex bodies directly in the same way as we do with extended support functions [4]. Moreover, in the finite-dimensional case computations are reduced to computations in a vector space.

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